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Kelvin–Helmholtz waves in the ocean?

By J. J. MAHONY

Department of Earth and Planetary Sciences, The Johns Hopkins University, Baltimore, Maryland 21218†

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Large amplitude short waves confined near the crests of a swell have been observed when a stiff breeze was blowing against the swell. This would seem to imply the existence of both a wavelength-selective generating mechanism and a trapping mechanism, neither of which is to be expected of surface gravity waves of the observed length. It is suggested that there are significant changes in the dynamics of such waves if allowance is made for the dynamic coupling between wind and waves. For a Kelvin–Helmholtz model it is shown that energy transfer rates from the turbulent pressure fluctuations are greatly enhanced for subcritical conditions by the inclusion of the dynamic coupling. The group velocity of subcritical waves is profoundly affected, becoming infinite at the stability boundary. Thus subcritical waves could be trapped on a swell. An examination of the effects of wind shear suggest that Kelvin–Helmholtz type instability could still be present, although for stronger winds, particularly for rather longer waves.

The energy and momentum fed from the mean wind, being trapped at crests of the swell, may contribute significantly to the attenuation of the swell. The profound wave dynamic effects of the coupling between the wind and the swell for short gravity waves may be of significance in other oceanic phenomena, even when the Kelvin–Helmholtz type of instability is not present.

1. Introduction

On two separate occasions during a recent sea voyage the author observed a persistent and striking surface wave phenomenon. On both occasions there was a moderate to heavy swell running with a stiff opposing breeze. In a zone approximately symmetrically located along each crest of the swell, quite steep waves with a wavelength of 0·1 m or somewhat larger could be seen. Over the remainder of the sea surface there was a remarkable absence of surface wave activity, apart from the swell itself, and the surface appeared smooth apart from having patches of residual foam. This foam originated from the breaking of the short waves grouped near the crests of the swell. Although there was considerable irregularity associated with the breaking of short waves the overall impression was of a regular pattern of short waves with their crests roughly parallel to the crests of the swell. Over a period of some hours, while the phenomenon persisted, there was a considerable abatement of the swell. In view of the energetic irreversible action occurring near each crest of the swell it is quite

† Permanent address: Department of Mathematics, University of Western Australia, Nedlands.

possible that the phenomenon described played an important role in attenuating the swell.

The comparative absence of short waves over most of the surface, together with the occurrence of zones of fairly regular intense short-wave activity, suggests that there are two actions for which an explanation is desirable. The first such action is a selective mechanism which confines the generation of waves by the wind to a relatively small band of wavenumbers. Second, there would appear to be acting some trapping mechanism which collects these waves near the crests of the swell. But if the local dynamics of these 0·1 m waves are those of surface gravity waves, as one would naturally expect, neither of these actions would appear to be possible. A stiff breeze would not be expected to generate selectively waves of this length but rather a broader spectrum centred on a somewhat larger wavelength. Even more striking is the fact that the group velocity of gravity waves of this length is an order of magnitude smaller than the velocity of the swell. One would therefore expect that the swell would sweep past these shorter waves. Further, the short waves are too long to feel the thin vortical film which can affect capillary waves as discussed by Phillips & Banner (1974).

One can only conclude that the local wave dynamics of these 0·1 m waves, under the conditions of the observations, are not those of surface gravity waves. Thus, apart from any interest in explaining the phenomenon itself, it would appear to be worth while to look more closely at the dynamics of short surface gravity waves in the presence of wind. The only physical action, of which the author could think, which could possibly produce such dramatic changes from the standard theory of surface gravity waves is the dynamic pressure variations induced in the air by the movement of the water surface. For waves of this length, the dynamic coupling between wind and waves will depend on the wind profile, which is likely to be modified by the presence of the swell. It seemed more appropriate as a first attempt to examine possible dynamic coupling between waves and wind to use a mathematically simpler, if less realistic, model. Thus in §§ 2 and 3 the modification of the local wave dynamics is investigated in the absence of wind shear. Of course, such a potential-flow model leads to the well-known Kelvin–Helmholtz theory, which is described in Lamb (1953). Such a model surely overestimates the possibility of Kelvin–Helmholtz instability. Nevertheless it is used in this paper for an initial investigation of a possible mechanism for the phenomenon observed by the author. The mathematical properties of a more realistic model would appear to be qualitatively similar.

Even allowing for the possible overestimate it is shown in § 2 that it is not implausible that Kelvin–Helmholtz instability occurs in the 0·1 m wavelength range. Moreover, near the stability boundary there is a major change in the behaviour of the group velocity which is not confined to a potential-flow model. Thus the inclusion of dynamic coupling with the wind leads to two features of local wave dynamics which are essential to explain the observed phenomenon.

In § 3 it is shown that there is a further feature of the model behaviour which may be of significance. It is shown that near, but below the stability boundary, the Phillips (1957) mechanism of energy transfer from turbulence is greatly enhanced. This mechanism, or rather the version associated with a more realistic wind shear model, could be significant in the transfer of energy to the very short surface waves: a process that appears to be poorly understood.

In § 4 the model is extended to allow for the presence of the swell. With wind and swell opposed the possibility arises that there will be a band of gravity wavenumbers which are unstable only near the crests of the swell and everywhere else are stable. In the troughs the energy is fed into these wavenumbers from the turbulence but because a relatively empty part of the turbulent pressure spectrum is involved the transfer rate is small. As the swell advances on these waves they come close to the stability boundary so their growth rate increases dramatically as does their group velocity. One wave family is trapped on the leading face of the swell but the other is swept onto the crest, where it gains energy directly from the wind and the rapid growth continues as it moves over the crest. It is shown that these waves are trapped on the rear face of the swell close to the point where they become stable again. If breaking of these waves occurs in this zone the resulting confused surface is likely to trap any shorter waves which have been generated. Thus since all the shorter waves are trapped near the crest, wave generation has to recommence in the trough below the station where trapping has occurred. The qualitative features of this model are thus completely consistent with the observations.

2. Kelvin–Helmholtz instability

The derivation of the dispersion relation for small amplitude waves on the interface between two fluids moving parallel to the interface with uniform speeds is well known: see, for example, Lamb (1953, p. **373**). If the velocity in the upper, lighter fluid is $\mathbf{U}'$ and that in the lower fluid is $\mathbf{U}$ then the natural angular frequencies ω of interfacial waves with wavenumber vector $\mathbf{k}$ are given by

$$\omega = (1+\Delta)^{-1}\{(\mathbf{U}+\Delta\mathbf{U}').\mathbf{k} \pm [(1-\Delta^2)\,gk + Tk^3\rho^{-1} - \Delta((\mathbf{U}'-\mathbf{U}).\mathbf{k})^2]^{\frac{1}{2}}\}. \tag{1}$$

Here T is the surface tension, g is the acceleration due to gravity, ρ is the density of the lower fluid and Δ is the ratio of the densities of the upper and lower fluids. For air and water Δ has a value around $1{\cdot}2\times 10^{-3}$ and hence it is reasonable to neglect many of the terms in (1). Throughout the remainder of the paper the dispersion relation will be taken to be of the form

$$\omega = \mathbf{U}.\mathbf{k} \pm [gk + Tk^3\rho^{-1} - \Delta\{(\mathbf{U}'-\mathbf{U}).\mathbf{k}\}^2]^{\frac{1}{2}}. \tag{2}$$

Such an approximation produces small errors in the parameters at which significant physical changes occur but leaves all basic phenomena well described. In the subsequent analysis some terms become extremely important because they involve the reciprocal of the difference of two nearly equal quantities. The author has checked that the above approximation does not invalidate the character of any of the subsequent analysis. In (2) the only additional term from the standard theory of surface waves, $\{(\mathbf{U}'-\mathbf{U}).\mathbf{k}\}^2\Delta$, represents the effect of the dynamic pressure changes induced in the air by the movement of the interface.

Complex values of ω arise, and hence the plane surface becomes unstable, for wavenumber vectors $\mathbf{k}$ such that

$$\Delta\{(\mathbf{U}'-\mathbf{U}).\mathbf{k}\}^2 \geqslant gk^{-1} + Tk\rho^{-1} = \{c(k)\}^2,$$

where c is the phase speed of gravity–capillary waves in the absence of wind. This phase speed has a minimum value at a wavenumber k_0 given by $k_0^2 = \rho g/T$, and the corresponding minimum wind speed W_0 for the occurrence of Kelvin–Helmholtz instability is given by

$$W_0 = \Delta^{-\frac{1}{2}} c(k_0).$$

For relative wind components W, defined by $W = (\mathbf{U}' - \mathbf{U}) . \hat{\mathbf{k}}$, greater than W_0 it can be shown that the range of wavenumbers for which Kelvin–Helmholtz instability can occur is given by

$$(W/W_0)^2 - \{(W/W_0)^4 - 1\}^{\frac{1}{2}} \leqslant k/k_0 \leqslant (W/W_0)^2 + \{(W/W_0)^4 - 1\}^{\frac{1}{2}}.$$

This range increases very rapidly with W/W_0, so that, while the wavelength at the critical wind speed of 6·7 m s^{-1} is only 17 mm, the largest unstable wavelength has increased to 0·14 m at a relative wind component of 13·4 m s^{-1}. For a relative wind component of 20 m s^{-1}, representative of strong but not excessive winds, the maximum wavelength is 0·33 m. In extreme storm conditions the instability could extend to waves almost one metre in length. It is apparent that for the stronger winds Kelvin–Helmholtz instability extends well into the gravity-wave range. The above results also indicate a quite strong angular dependence of the effect because for wavenumber vectors not aligned with the relative wind the ratio W/W_0 is reduced and the longer wavelengths are not unstable.

It is also of interest to note how large the growth rates $\gamma(\mathbf{k})$, given by

$$\gamma(\mathbf{k}) = c(k_0)\, k_0 \{(W/W_0)^2 (k/k_0)^2 - \tfrac{1}{2}(k/k_0) - \tfrac{1}{2}(k/k_0)^3\}^{\frac{1}{2}}, \tag{3}$$

can be. If W/W_0 is as much as 2, then at a wavenumber $k_0(W_0/W)^2$, which is well inside the gravity-wave range,

$$\gamma(k) \approx c(k_0)\, k_0/\sqrt{2}.$$

This gives as a representative e-folding time, for gravity waves in moderately strong winds, the value 0·03 s. In the capillary wavenumber range the growth rates are very much larger again but in view of the fact that a potential-theory model is entirely inadequate for such waves this is probably irrelevant under, say, severe storm conditions.

Hereafter attention will be confined to wavelengths in the gravity-wave range, for which it is reasonable to neglect surface tension. This is done for analytic convenience and does not affect the qualitative conclusions reached. But it is not only in the introduction of an instability that the dynamic coupling has important effects on the short-wave dynamics. For wavenumbers outside the instability band, but not too far divorced from it, there is a most significant modification of the dispersion relation. One quite dramatic effect appears in the group velocity and a second will be taken up in the next section. The group velocity is given by

$$\nabla_k \omega = \mathbf{U} \pm \tfrac{1}{2}\{g\hat{\mathbf{k}} - 2\Delta W k(\mathbf{U}' - \mathbf{U})\}\{gk - \Delta W^2 k^2\}^{-\frac{1}{2}}$$

when $gk > \Delta W^2 k^2$. If one confines attention to the case, suggested above as being of the most interest, when $\mathbf{k}$ is parallel to the relative wind, the discussion becomes a little easier. For wavenumbers such that $\Delta W^2 k/g$ is small the group velocity relative to the fluid is little affected and is equal to half the phase velocity. When $\Delta W^2 k/g$ has grown to a half, the group velocity relative to the fluid vanishes. For larger values the group velocity changes rapidly with wavenumber, or relative wind speed, and becomes unbounded at the stability boundary. The fact that short waves can have group velocities large in comparison with their phase velocities opens up the possibility that they could be trapped on the very much longer waves of a swell. This effect, very much localized in wavenumber space, could thus be of considerable physical importance.

3. Energy transfer to surface waves

In this section two quite distinct mechanisms of energy transfer into water-wave motion will be investigated for waves for which dynamic coupling could be important. It will be assumed that the motion of both air and water due to the movement of the interface can be calculated using potential theory. Nevertheless it will be assumed that the otherwise uniform wind carries turbulent pressure fluctuations which, acting on the interface, initiate the surface movement. The model is thus an extension of the one proposed by Phillips (1957). Thus a solution is sought in terms of a surface displacement

$$\zeta = A(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}},$$

a velocity potential

$$\phi' = B'(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}-kz}$$

in the air and a velocity potential

$$\phi = B(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}+kz}$$

in the water. In the air it is assumed that the pressure is the sum of the pressure calculated from ϕ', using Bernoulli's principle, and a turbulent pressure component $\rho p(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x}), \rho$ being the density of water. In the water it is assumed that there is no turbulent pressure component. The model thus ignores the wind shear necessary for the maintenance of the turbulence and any coupling between the surface-generated pressure field and the turbulence.

The functions A, B and B' can be determined in principle from the interface conditions. The two kinematic conditions yield

$$\{\partial/\partial t + i\mathbf{k}.\mathbf{U}'\}A = -kB'$$

and

$$\{\partial/\partial t + i\mathbf{k}.\mathbf{U}\}A = kB$$

while the pressure condition yields

$$\{\partial/\partial t + i\mathbf{k}.\mathbf{U}\}B + gA = p + \Delta[\{\partial/\partial t + i\mathbf{k}.\mathbf{U}'\}B' + gA].$$

Elimination of B and B' from these three equations, together with the use of the approximation Δ small in the manner of the derivation of (2) from (1), yields

$$(\partial/\partial t + i\mathbf{k}.\mathbf{U})^2 A + \{gk - \Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2\}A = kp \tag{4}$$

for the equation governing the surface response. The subsequent calculations could be made without using Δ small but with no essential change. It is now convenient to consider quite separately the cases when the wavenumber vector $\mathbf{k}$ is stable or unstable.

Consider first the case of unstable wavenumbers. Referred to axes moving with the water (4) becomes

$$\partial^2 A/\partial t^2 - \gamma^2 A = kp'(\mathbf{k},t), \tag{5}$$

where p' is the turbulent pressure Fourier component relative to these moving axes. Now the turbulent pressure fluctuations are most highly correlated for points moving with the wind speed, i.e. $\mathbf{U}'-\mathbf{U}$ relative to these new axes. This implies that the time scale associated with the fluctuations of p' is $\{(\mathbf{U}'-\mathbf{U}).\mathbf{k}\}^{-1}$ while that associated with the growth by Kelvin–Helmholtz instability is larger by a factor of at least $\Delta^{-\frac{1}{2}}$, which is about 35. The solution of the differential equation (5) satisfying zero initial conditions is

$$A = k\gamma^{-1}\int_0^A p'(\mathbf{k},s)\sinh\gamma(t-s)\,ds$$

and the portion which is growing in time is given by

$$A_+ = k(2\gamma)^{-1}e^{\gamma t}\int_0^t p'(\mathbf{k},s)\,e^{-\gamma s}\,ds.$$

The estimation of this term causes considerable difficulty because of the different orders of the two time scales, noted above, and the fact that p' has zero mean.

One can note that A_+ satisfies the differential equation

$$\partial A_+/\partial t-\gamma A_+ = k(2\gamma)^{-1}p',$$

and this admits an iterated solution

$$A_+ = k(2\gamma)^{-1}\sum_{n=0}^{\infty}\gamma^n I^{(n+1)}(p'),$$

where I is the integral operator defined by

$$I(p') = \int_0^t p'(\mathbf{k},s)\,ds.$$

Because p' has zero mean $I(p')$ does not grow with t but as $I(p')$ has non-zero mean $I^2(p')$ will eventually grow as $t^{\frac{1}{2}}$, a standard result for a random walk. Hence $I^{n+2}(p')$ will behave as $t^{n+\frac{1}{2}}$ once the trend has been established with I^2. Thus one can expect exponential growth to occur once some trend has been established by the early-stage pressure fluctuations. Subsequent turbulent pressure fluctuations contribute to A to a much lesser extent and hence the eventual response A_+ is in fact given by the homogeneous equation

$$\partial A_+/\partial t-\gamma A_+ = 0$$

together with an initial condition determined by the feed from the early stages of the pressure fluctuations. Once the instability has been initiated the turbulence is no longer the source of energy for the growth of the waves. An energy calculation shows that the surface wave energy is equal to the kinetic energy of the wind in the zone which is, on average, screened by the surface undulations. This clearly implies that for the unstable wavenumbers the surface waves are fed from the mean wind. The turbulence merely initiates the instability and thereafter has no role to play in their energy supply. It may also be noted that the turbulence is not a particularly efficient initiator and there is a delay before the growth is likely to start. As will be shown later, it is likely that another action will provide the initiation.

For the stable wavenumbers it is convenient to recognize that the surface response is likely to be largely stochastic but that there may be a slow rate of change in the level of that response due to energy transfer of the type considered by Phillips (1957). Thus one seeks to describe the function by a representation

$$A(\mathbf{k},t) = \int_{-\infty}^{\infty} \bar{A}(\mathbf{k},\omega,t)\, e^{-i\omega t}\, d\omega. \tag{6}$$

Here the time dependence in $\bar{A}$ is used to describe a slow rate of change in the general level and is thus not to be included in any statistical averaging process. This distinction in roles implies that $|\partial \bar{A}/\partial t| \ll |\omega \bar{A}|$ and suggests using approximations in which $\omega^{-1}\partial/\partial t$ is regarded as a 'small operator'. Thus when the representation (6) is substituted in (4) and second-order terms in the slow trend are ignored the differential equation

$$2i(\mathbf{U}.\mathbf{k}-\omega)\,\partial\bar{A}/\partial t + \{gk - \Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2 - (\omega - \mathbf{U}.\mathbf{k})^2\}\bar{A} = k\bar{p}(\mathbf{k},\omega) \tag{7}$$

is obtained. Here $\bar{p}$ is defined by

$$p(\mathbf{k},\ t) = \int_{-\infty}^{\infty} \bar{p}(\mathbf{k},\omega)\, e^{i\omega t}\, d\omega$$

since one is not concerned with changes in the level of the turbulent pressure fluctuations. The solution of the differential equation with zero initial value is

$$\bar{A} = -k\bar{p}(1-e^{-i\sigma t})\,[2\sigma(\omega - \mathbf{U}.\mathbf{k})]^{-1},$$

where

$$\sigma = \tfrac{1}{2}\{gk - \Delta\,[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2 - (\omega-\mathbf{U}.\mathbf{k})^2\}\,(\omega - \mathbf{U}.\mathbf{k})^{-1}.$$

The mean-square expectation $\langle AA^*\rangle$ is given by

$$\begin{aligned}\langle AA^*\rangle &= (2\pi)^{-1}\int_{-\infty}^{\infty} \bar{A}\bar{A}^*\, d\omega \\ &= (4\pi)^{-1}k^2 \int_{-\infty}^{\infty} \Pi(\mathbf{k},\omega)\,\sigma^{-2}(\omega-\mathbf{U}.\mathbf{k})^{-2}\,(1-\cos\sigma\ t)\,d\omega,\end{aligned}$$

where

$$\Pi(\mathbf{k},\omega) = \bar{p}(\mathbf{k},\omega)\,\bar{p}^*(\mathbf{k},\omega).$$

Thus the rate of increase of the mean-square expectation, which is closely related to the energy density in the wavenumber spectrum, is given by

$$\frac{d}{dt}\langle AA^*\rangle = (4\pi)^{-1}k^2\int_{-\infty}^{\infty} \Pi(\mathbf{k},\omega)\,(\omega-\mathbf{U}.\mathbf{k})^{-2}\sigma^{-1}\sin\sigma t\, d\omega.$$

The pole at $\omega = \mathbf{U}.\mathbf{k}$ is not of concern because there σ is infinite and, while the integrand is unbounded, the neighbourhood of the point contributes negligibly to the integral. Now t has been included, without any scale change, to represent changes taking place on a time scale large in comparison with ω^{-1}. Thus it is essentially the asymptotic value for large values of t which is of interest. The Riemann–Lebesgue lemma and stationary-phase estimates show that most of the range of integration does not contribute significantly for large values of t. However the neighbourhoods of those values of ω for which $\sigma = 0$ will contribute non-vanishing terms in the limit $t \to \infty$. There are two such values ω:

$$\omega = \mathbf{U}.\mathbf{k} \pm \{gk - \Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2\}^{\frac{1}{2}} = \mathbf{U}.\mathbf{k} \pm \Omega(\mathbf{k}),$$

corresponding to the natural frequencies of interface waves with the wavenumber vector $\mathbf{k}$. It is of interest to note that if $\mathbf{k}$ lies in the domain of Kelvin–Helmholtz instability there are no such values of ω and there is no energy feed from the turbulence within the accuracy of the approximations made. This is in accord with the conclusion reached earlier that it is the mean flow, and not the turbulence, which feeds the growth of that instability. For a stable wavenumber a routine calculation yields

$$\frac{d}{dt}\langle AA^*\rangle = (4\pi)^{-1}k^2\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds\,\{gk-\Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2\}^{-1} \times\{\Pi(\mathbf{k},\mathbf{U}.\mathbf{k}-\Omega(\mathbf{k})+\Pi(\mathbf{k},\mathbf{U}.\mathbf{k}+\Omega(\mathbf{k}))\}. \quad (8)$$

This is essentially an alternative derivation of Phillips' (1957) result for surface waves allowing for the modification of the dispersion relation due to coupling with the wind. The result applies only for those wavenumber vectors for which $\{gk-\Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2\}$ is positive and hence the energy transfer is always from the turbulence to the waves. But the most striking feature of the result is the fact that, for wavenumbers close to the stability boundary, there is a greatly increased response because

$$\{gk-\Delta[(\mathbf{U}'-\mathbf{U}).\mathbf{k}]^2\}$$

will be small. It must be borne in mind that $\Pi(\mathbf{k},\omega)$ has its maximum for $\omega=\mathbf{U}'.\mathbf{k}$ and decays away from that value, so that its level in the neighbourhood of $\mathbf{U}.\mathbf{k}$ is likely to be rather small. The author knows of no way to obtain a reasonable estimate of this term, which would involve a spatial and temporal Fourier analysis of the turbulent pressure field over the ocean. But for a narrow band of wavenumbers close to the stability boundary there may be significant energy transfer. Certainly there is the possibility of a wavelength-selective transfer mechanism, which is not possible without the inclusion of the dynamic coupling. One may also comment that the smaller the value of $\Pi(\mathbf{k},\mathbf{U}.\mathbf{k})$ the narrower will be the band of wavenumbers for which the enhanced growth rate will be significant. Further, it would appear that there is no reason why a qualitatively similar analysis should not apply near the stability boundary for some other systems. In particular one would expect that if the effects of wind shear were included γ^2 in (5) would be replaced by the term deriving from the modified dispersion relation and only algebraic details would be changed.

It may be of interest to note that the unbounded energy transfer rate is not an integrable function of k, so that the actual singularity at the critical boundary should not be taken too seriously. The above theory assumes that the turbulent pressure spectrum is given and is unaffected by any energy transfer to the surface waves. It of course cannot be true that the turbulent spectrum is unaffected when large energy transfers can occur in a limited wavenumber range. There would appear to be little chance of making theoretical predictions which include the effect of energy transfer to the waves on the turbulence itself. Nevertheless, as will be seen later the effect of the swell may be such as to make such calculations less necessary.

4. Short waves on a swell

The previous two sections have been concerned with the local dynamics and generation of short waves under uniform conditions. When a swell is present both $\mathbf{U}'$ and $\mathbf{U}$ will vary in both space and time and in this section the question of the cumulative effect of variation of local conditions due to the presence of a swell will be investigated. The potential-theory model will still be used.

The presence of a swell produces a number of effects which will modify the local dynamics. The relative wind velocity $\mathbf{U}'-\mathbf{U}$ will vary with position relative to the swell crests and hence on length and time scales which are large in comparison with those of the short waves in the neighbourhood of the stability boundary. Further, the vertical acceleration of the swell provides a similar slow variation of the effective acceleration due to gravity. Thus a given wavenumber vector may be stable on one part of the swell and unstable on another part. In addition, there are convergence–divergence effects associated with the swell which will modify the propagation of the short waves. All these features must be included in the calculations. Given the inadequacies of a potential-flow model and the argument that the largest overall effect is most likely to occur when the individual effects are all aligned, only a one-dimensional model will be investigated here. Further, it will be assumed, in the sign conventions adopted, that the wind and swell are opposed.

A routine calculation shows that the surface velocity, correct to first order in the surface slope, produced by a wind of speed W blowing over a moving surface

$$y = a\cos K(x+Vt)$$

is given by

$$U' = W+\alpha(V+W)\cos K(x+Vt),$$

where α, equal to Ka, is the maximum slope of the surface due to the swell. Here W and V are both positive for opposing wind and swell. Similarly, the water velocity due to the swell is given by

$$U = -\alpha V\cos K(x+Vt),$$

so that the relative wind is given by

$$U'-U = W+\alpha(2V+W)\cos K(x+Vt).$$

Except in storm conditions, the swell velocity is likely to be of the order of two or three times the wind speed, so that there are considerable variations in the relative wind speed between the crests and troughs. The extent of these variations depends on the swell slope but heavy swells will have remarkably large variations. For opposing wind and swell the relative wind is greatest at the crests and least at the troughs. This variation is reversed when the wind and swell are aligned. Further, the effective gravity is $(1-\alpha)g$ at a crest and $(1+\alpha)g$ at a trough. Thus for opposing wind and swell the variations act together to increase the wavelengths which can be unstable near the crests and to reduce the critical wavelength near the troughs. Thus the largest wavelengths which are anywhere unstable will be unstable near the crests and stable everywhere else.

This is pointing in the direction of the phenomenon under investigation and it is of interest to substitute some hypothetical numerical values to see the size of the variations involved. These values are not to be taken as related to any specific observation. For a slight swell of wavelength 400 m and amplitude 2 m the relative wind would vary from 8 m s^{-1} at the troughs to 12 m s^{-1} at the crests for a stiff breeze of 10 m s^{-1}. The calculations of § 2 show that waves of length 42 mm will be unstable at a trough and waves of length 100 mm at a crest. For a swell amplitude of 4 m no wavelength is unstable at a trough and the critical value at a crest is about 0·16 m. It can be seen that when strong winds and heavy seas are involved there are quite dramatic shifts in the stability condition between trough and crest. When the wind and sea are running in the same direction it is the troughs where the instability is most likely but the variations are not so marked.

On the assumption that the Kelvin–Helmholtz type of instability will be reduced by the wind shear, attention will be confined to the larger wavelengths which are unstable only near the crests. At first the generation of these waves and their propagation in the zone where they are stable are considered. These waves are short waves moving on the surface of the swell and it is not unreasonable to describe them by a linear theory of slowly varying waves. While there are a number of general discussions of such a theory, e.g. Bretherton & Garrett (1969), there are some special features in the present context. For short waves of say 0·1 m on a swell of length 400 m the ratio of the length scales K/k is $2{\cdot}5 \times 10^{-4}$ but the ratio of the time scales is $VK/(Uk)$ and this is larger by a factor α^{-1}. In the phenomenon this paper sets out to explain the short waves are virtually absent in the troughs of the swell, so that it is perfectly reasonable to presume that the accumulation over one swell period is all that need be considered. As one is not contemplating swells with α larger than about 0·03 this therefore suggests neglecting terms in the slowly varying dynamics which are either $O(\alpha^2)$ or $O(K/k)$. Neglecting terms $O(\alpha^2)$ is the more dubious assumption on grounds of size, but it does not vary the physics and as the theory is not expected to be particularly accurate there seems little point in retaining such terms. Neglect of terms $O(K/k)$ is much more justified on numerical grounds but involves the neglect of the group velocity. As the group velocity becomes large close to the stability boundary the present development will need modification there. It is important to recognize one feature of the present problem. In the velocity field U on which the waves are moving the acceleration U_t rather than the velocity gradient U_x is the major variation the short waves feel. This is a consequence of the above observation about slow length and time scales. Thus one should not expect the results of Bretherton & Garrett to apply. In their discussion they specifically exclude the case where the rate of change of particle velocities in the base motion is not of at most second order in the short-wave amplitude. Thus a brief outline of the derivation of the appropriate equations will be given here.

The appropriate linearized kinematic interface condition for the short waves on the swell is obtained from the standard condition

$$Z_t + \Phi_x Z_x = \Phi_y \quad \text{on} \quad y = Z(x,t)$$

by taking the Frechet derivative. Thus one obtains

$$\zeta_t + \Phi_x \zeta_x + \phi_x Z_x = \phi_y + \Phi_{yy}\zeta \quad \text{on} \quad y = Z.$$

Here the lower case symbols denote the surface elevation ζ and velocity potential ϕ due to the short waves and the upper case symbols those due to the swell. Now $\Phi_{yy}\zeta$, being smaller than $\Phi_x \zeta_x$ by a factor $O(K/k)$, can be neglected. Thus this boundary condition reduces to

$$(\partial/\partial t + U\partial/\partial x)\,\zeta = \phi_y - \phi_x Z_x \quad \text{on} \quad y = Z,$$

where U is the surface velocity due to the swell, and the corresponding condition for the air motion is

$$\left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)\zeta = \phi'_y - \phi'_x Z_x \quad \text{on} \quad y = Z.$$

It can be shown that the pressure condition yields

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi+(g+\Phi_{yt})\zeta+\Phi_y\phi_y = P(x,t)+\Delta\left(\frac{\partial}{\partial t}+U'\frac{\partial}{\partial x}\right)\phi',$$

where ρP is the turbulent pressure and advantage has been taken of the smallness of Δ. It is convenient to introduce a new co-ordinate η, defined by $\eta = y-Z(x,t)$, so that the boundary conditions can be applied at a fixed value of one co-ordinate. The modified boundary conditions become

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\zeta = \phi_\eta-\phi_x Z_x,$$

$$\left(\frac{\partial}{\partial t}+U'\frac{\partial}{\partial x}\right)\zeta = \phi_\eta-\phi'_x\ Z_x,$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi+(g+Z_{tt})\zeta = P+\Delta\left(\frac{\partial}{\partial t}+U'\frac{\partial}{\partial x}\right)\phi',$$

to be satisfied on $\eta = 0$. Use has been made of the condition $\Phi_y = Z_t$. In the new co-ordinate system, when terms $O(\alpha^2)$ are neglected Laplace's equation becomes

$$\phi_{\eta\eta}+\phi_{xx}+2\phi_{x\eta}Z_x = 0, \tag{9}$$

and solutions are sought in the form

$$\zeta = A(k,t,X)\,e^{ikx},\quad \phi = B(k,t,X,\eta)\,e^{ikx},\quad \phi' = B'(k,t,X,\eta)\,e^{ikx},$$

where

$$X = Kx.$$

If (9) is solved to determine ϕ and ϕ' in terms of the values of B and B' at $\eta = 0$ and these values are then eliminated from the boundary conditions, it can be shown that

$$\left\{\left(\frac{\partial}{\partial t}+iU\mathscr{K}\right)\mathscr{K}^{-1}\left(\frac{\partial}{\partial t}+iU\mathscr{K}\right)\right\}A+\left\{\Delta\left(\frac{\partial}{\partial t}+iU'\mathscr{K}\right)\mathscr{K}^{-1}\left(\frac{\partial}{\partial t}+iU'\mathscr{K}\right)+(g+Z_{tt})\right\}A = p(k,t), \tag{10}$$

where $\mathscr{K}$ is an operator defined by

$$\mathscr{K} = k-iK\partial/\partial X.$$

In view of the estimates obtained previously of the relative scales of time and length which are of interest it is reasonable to neglect all derivatives with respect to X occurring in (10). The parameters defining the effects of the swell are all functions of the single combination

$$\xi = X+KVt$$

and it is convenient to use this fact to define a new independent variable

$$B = A\exp\left\{ik\int_{\xi_0}^{\xi}(U+\Delta U')/\{(1+\Delta)\,KV\right\} = A\exp\{iS(\xi)\},$$

where S is defined by this equation and ξ_0 is some reference point on the swell. Then (10) may be simplified to

$$(1+\Delta)\frac{\partial^2 B}{\partial t^2}+\{(g+Z_{tt})\,k-\Delta(U'-U)^2k^2/(1+\Delta)\}\,B = kp(k,t)\,e^{iS(\xi)}.$$

It may be observed that A and B differ only in terms of phase and that the gradient of this phase factor corresponds locally to changing to a frame of reference moving with the mean speed of the local surface wave modes when dynamic coupling is considered. In particular it implies that the expected wave energy may be computed directly from B. One may further note that, in the absence of turbulent pressure fluctuations for the calculation of the development of short waves on the swell, Green's formula would apply and that one would expect the short-wave amplitude to grow as the station where the stability boundary for the given value of k is approached.

In a theory of the limited attempted accuracy of this paper there is little point in retaining terms genuinely $O(\Delta)$ and so the equation for B may be reduced to the form

$$\partial^2 B/\partial t^2 + \chi(k,\xi)\,B \approx kp'(k,t,\xi)$$

$$= kp(k,t)\,e^{iS(\xi)}, \tag{11}$$

where $$\chi = (g+Z_{tt})\,k - \Delta(U'-U)^2 k^2,$$

in line with previous approximations for the Kelvin–Helmholtz stability boundary. In the present context only wavenumbers k which are such that χ is negative at crests, i.e. wavenumbers unstable at crests, are of interest. Further, it will be assumed here, and explained later, that no short waves are left behind after the zone near the crest has swept by, as was observed. Thus it is convenient to choose a value of ξ_0 ($> -2\pi$) such that B vanishes at this point. It will then suffice to explain the observed distribution of waves and the use of the initial condition will be shown to be self-consistent if it can be established that $B(k, \xi_0 + 2\pi)$ vanishes.

Let ξ_1 denote the smallest negative value of ξ at which χ vanishes for the particular wavenumber. Then on the basis of the spatially homogeneous theory it is appropriate to seek solutions for $\xi_0 \leqslant \xi < \xi_1$ of the form

$$B = \int_{-\infty}^{\infty} \bar{B}(k,\omega,\xi)\,e^{-i\omega t}\,d\omega$$

since ξ may equally well be used as a slow time variable. A standard application of multiple-scale arguments leads to the equation

$$-2i\omega K V\,\partial\bar{B}/\partial\xi + (\chi-\omega^2)\,\bar{B} = k\bar{p}'(k,\omega,\xi),$$

which may be solved. The pattern of calculation followed in § 3 then leads to the result

$$\langle AA^*\rangle(\xi) \propto k^2 \int_{-\infty}^{\infty} \omega^{-2}\,\Pi(k,\omega-Uk) \int_{\xi_0}^{\xi}\int_{\xi_0}^{\xi} \exp\left[i\int_{\xi''}^{\xi'} \frac{\chi-\omega^2}{2\omega K V}\right] d\xi'\,d\xi''\,d\omega. \tag{12}$$

This result applies only if it is valid to calculate expected energies by integrating over time intervals small in comparison with the period of the swell. It is almost certain that such a method will not lead to particularly accurate results. But since little is known about the turbulent spectrum there would appear to be little point in seeking a better estimate. But is should be borne in mind that the effect of the stochastic processes is to reduce the rate of energy transfer, so that the present approach would be likely to underestimate the wave energy. Thus the integrals in (12) will be estimated by asymptotic methods. The integrand will be highly oscillatory save when ξ'' is close to ξ' and there

$$\int_{\xi_0}^{\xi} \exp\left[i\int_{\xi''}^{\xi'} (\chi-\omega^2)/(2\omega KV)\right] d\xi'' \approx \int_{-\infty}^{\infty} \exp\left[-i(\xi''-\xi')\,\frac{\chi(\xi')-\omega^2}{2\omega KV}\right] d\xi''$$
$$= \delta(\omega-\chi/\omega).$$

Thus

$$\langle AA^*\rangle(\xi) \propto k^2 \int_{\xi_0}^{\xi} \chi^{-1}\{\Pi(k,\,-Uk-\chi^{\frac{1}{2}})+\Pi(k,\,-Uk+\chi^{\frac{1}{2}}\}\,d\xi',$$

which may be readily interpreted as the integral of the homogeneous transfer rate as calculated from local conditions.

Again it may be remarked that these integrals do not exist in the limit as ξ_1 is approached. But in this case it is not only the inability of the turbulence to supply energy at the required rate which will serve to invalidate the model. There will also be large spatial gradients developing. Hence the assumption that K^{-1} is the appropriate slow length scale will not be reasonable and a further modification would be necessary near ξ_1. Away from this zone, however, the calculations would seem appropriate. But when χ is not small one expects the local transfer rate to be extremely small because in such frequency ranges Π should be very small. In these circumstances it is not hard to understand that there should be very little short-wave generation over the majority of the swell and that it is when one is relatively close to the critical zone for the given wavenumber k that quite rapid growth might be expected.

However as this zone is approached the assumption that the ratio of length scales is $O(K/k)$ will fail. It does not appear worth while to attempt detailed calculations of the behaviour in this zone given all the uncertainties. Rather it seems appropriate to use physical arguments to understand what is likely to happen in such zones. It is well understood, see, for example, Phillips (1966, p. 57) and Bretherton & Garrett (1969), that in computing the changes in any characteristics of waves it is appropriate to integrate along paths $dx/dt = u+g$, where u is the local fluid velocity and g is the group velocity.

There are two families of waves with frequencies

$$\omega = Uk \pm \{g'k - \Delta(U'-U)^2 k^2\}^{\frac{1}{2}}$$

respectively. Consider the 'plus family' of waves. Their group velocity becomes large and negative in a neighbourhood of a critical zone. Thus the paths where the group velocity equals the swell velocity will be fixed relative to the swell and paths on either side of this point and close to it move towards this path. Thus the plus family of waves will be trapped near this station. Although the earlier calculations do not necessarily apply these waves will presumably absorb energy from the turbulence, so that the amplitude of the plus family may be expected to grow large in this trapping zone. The 'minus family' of waves behaves quite differently. Their group velocity becomes large and positive, so that the passage of these waves through the zone is accelerated and they enter the zone where the Kelvin–Helmholtz instability is initiated.

Once the minus family is in the unstable zone it is reasonable to infer from the theory in § 3 that the turbulent pressure fluctuations will play little further role in the growth of the short waves. Then provided that it remains valid to neglect the spatial derivatives (11) reduces to

$$(KV)^2 d^2B/d\xi^2 + \{\chi(\xi) - \omega^2\} B = 0. \tag{13}$$

It is reasonable to assume that there is a sufficient level of disturbance at $\xi = \xi_1$ to initiate the Kelvin–Helmholtz instability immediately. In the unstable zone

$$\Delta(U' - U)^2 k^2$$

is of the same order as gk, and as V^2 is equal to gk, the growth rate in terms of the dimensionless variable ξ is $O[(k/K)^{\frac{1}{2}}]$, which is about 10^2. Thus only a very small initial value and a narrow zone of instability on the crest would seem necessary in order that short waves originating in the minus family should be strongly excited near the crests. The above growth rate is so large that one can scarcely justify the neglect of spatial derivatives in the deduction of (11). But in the observations the short waves were seen to be breaking throughout the zone where there was intense short-wave activity. This would suggest that the extent of excitation is the result of a balance between the energy transfer from the mean wind and that lost owing to breaking. In such circumstances it would not appear to be useful to seek a modification of (13) to include the neglected spatial derivatives without at the same time including some representation of breaking.

In the zone of instability the swell merely sweeps past the growing short waves and one is thus interested in knowing what is the appropriate behaviour of the solutions of (10) for $\xi > \xi_1$. A complete discussion would need to include appropriate group-velocity effects near the stability boundary, breaking and feed from the turbulence for ξ near $\pm\xi_1$. This looks impossible and so it seemed appropriate to work with (13) together with the initial condition that it is the minus family of waves which enter the unstable zone. The question is thus reduced to finding the behaviour of a solution of (13) in the region $\xi > \xi_1$ which behaves largely like

$$\exp\left(-i\int^{\xi} \chi^{\frac{1}{2}}(KV)^{-1}\right)$$

for $\xi < -\xi_1$. Methods of doing this are described in Heading (1962) but care is necessary in translating his results to the present problem. For Heading is concerned with wave reflexion problems where there is energy conservation in the waves, whereas here the waves are deriving energy from the wind and wave reflexion is not acceptable for it would imply that for $\xi < -\xi_1$ the short waves would be running ahead of the swell. However Heading's connexion formula can be combined to give the appropriate result. When $\xi > -\xi_1$ the above solution has two parts one of which grows exponentially and the other of which decays exponentially and can be expected to be negligibly small when ξ is near ξ_1. The exponentially large solution connects with a solution whose wave propagation behaviour is given by

$$\exp\left(i\int^{\xi} \chi^{\frac{1}{2}}(KV)^{-1}\right)$$

and this has a group velocity which is large and negative. Thus the waves which have grown exponentially as the crest has swept past will be trapped on its rear face close to the point where they have become stable. The qualitative explanation of the observed confinement of the short-wave activity to zones close to the crest is thus satisfactory.

5. Conclusions

There are no solid predictions in the theory, as it has been developed, which could be subjected to critical testing by comparison with either experiment or observation. Nevertheless the author believes that the general qualitative explanation of the observations suggests that the basic hypothesis, that dynamic coupling between the wind and the surface waves may be important, is at least worthy of further investigation. Certainly none of the features of the observed localization of the short waves near the crests of the swell are easily explained from the local dynamics normally assumed to apply for short waves. What needs to be established is whether the processes considered here are in fact sufficiently powerful to give rise to sufficient energy transfer. In order to do this there are improvements of the theory which it would be essential to achieve.

By far the most serious failing of the present model is the persistent use of potential-flow theory to compute the dynamic coupling. A more realistic calculation must allow for the very significant shear in any wind near the surface of the ocean. It is obvious that the fact that the wind velocity is reduced close to the surface will lead to a smaller coupling coefficient than that predicted using potential theory. Whether this reduction would be so great as to render impossible the mechanisms considered here must remain a matter of speculation. The author is inclined to the belief that the observed phenomena suggest that the mechanisms are not rendered ineffective. To test this he proposes to calculate a dispersion relation for surface waves more appropriate for shear flows. If such a relation can be obtained and there is evidence of Kelvin–Helmholtz type instability or a close approach thereto it would seem appropriate then to attempt a more careful analysis of the interaction with the swell.

There is one further feature of the trapping of short waves near the crests of a swell in an opposing wind which calls for comment. The energy of the waves derives from the wind and when breaking occurs some of this energy may be dissipated. But the wind is also supplying momentum to the waves in the direction of the wind. This momentum is not lost in the breaking process but must be redistributed in the subsequent fluid motion. When it is recognized that there are similar momentum sources located in zones about the crests of a swell one sees that there is a spatial periodicity. Hence one would expect that a not inconsiderable fraction of the momentum would be redistributed with the spatial periodicity of the swell, i.e. as waves like the swell. But as the momentum is opposed to that of the swell one would expect the spatially periodic distribution of momentum sources moving with the swell to act so as to reduce the swell. The observed phenomenon may thus play an important role in the action of strong opposing winds in the attentuation of a heavy swell.

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